X-553-71-106

PREPRINT

NASA TM X- 65646

A COMPARISON OF COWELL VS POWER SERIES NUMERICAL INTEGRATION AS APPLIED TO ORBITAL CALCULATIONS

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MARCH 1971

GSFC — GODDARD SPACE FLIGHT CENTER —
GREENBELT, MARYLAND

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ABSTRACT

This paper presents the results of a comparison of the classical Cowell numerical integration method with the power series integration method as applied to artificial satellite orbital calculations. These integration methods are described in detail, the relative advantages and disadvantages of each are discussed, and numerical examples are given. The conclusion derived from this study is that if the orbital motion is governed by a simple, smooth force model, the power series method is most effective; however, for applications involving complex or non-smooth models, such as those occurring in definitive near-Earth trajectory calculations, the Cowell technique is more efficient. For example, the power series method is shown to be 10 to 15 times more expensive than the Cowell method when used to integrate the motion of an Earth satellite such as GEOS-B; yet both methods achieve equivalent accuracies.

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A COMPARISON OF COWELL VS POWER SERIES NUMERICAL INTEGRATION AS APPLIED TO ORBITAL CALCULATIONS

INTRODUCTION

The following is a report on a comparison study between two numerical integration methods used for the integration of the equations of motion of a satellite and the concomitant variational equations. The first is the classical "second sum" Cowell method based on an integrated Newtonian interpolation polynomial and the second method involves a recurrent power series time expansion of the acceleration model itself. Both these methods are well known. The Cowell method is used in many of our current orbit determination programs and is derived in detail in Henrici [1]. The power series method, although not as widely used, is also well known for its applications in celestial mechanics. For example, Steffensen [2] applies the method to the restricted three body problem. A similar, more recent analysis is given by Fehlberg [3]. The purpose of this study is to determine the relative effectiveness of the power series technique as applied to the integration of the equations of motion of an artificial satellite disturbed by a non-spherical primary and other effects, and also to the integration of the variational equations required to form the state transition matrix.

COMPARISON CRITERIA

Generally, the simplest comparison criterion for different interpolatory or polynomial type numerical integration procedures is the number of function evaluations required to achieve a demanded accuracy for a fixed integration interval. This criterion is valid for all Runge-Kutta, Adams, Nordsieck, Cowell etc. methods, in situations where the effort required to evaluate the derivative dominates the computational effort of the method itself. For applications to the accurate computation of satellite trajectories in the vicinity of a non-spherical body, this is always the case. This criterion, as opposed to direct measurement of computation time, enjoys the property that it does not depend on computer anomalies such as core speed, compiler quality, programmer's skill, etc. Unfortunately, this criterion is not valid for the power series method since at each integration step, the function being integrated is evaluated together with its derivatives which generally significantly increases the computational effort. Furthermore, in the case of orbital motion, these derivatives can be computed recurrently, making the actual computational cost per step for the method difficult to calculate in terms of function evaluations. For these reasons, the basic criterion for comparison used in this report is the amount of

computer time required by each of the methods to achieve a specified orbital accuracy in the position and position partials after a given period of integration. In each case an effort is made to show that the most efficient mode of operation available was used for each method.

Hence, unavoidably, our numerical comparison results will not only reflect the relative merits of the integration methods, but also the computer programs used. To try to minimize any bias, the programs used were those written by the developers of the particular techniques as applied to the problem (references [4] and [5]), the problem parameters (orbit, force model, etc.) and integration parameters (stepsize, order, tolerances, etc.) were varied to demonstrate the performance of each method under different conditions, and the reference solutions used to measure accuracy were obtained by running each respective program in a high accuracy mode.

Other considerations which should be made when comparing integration methods is the relative ease in which each method can be implemented and maintained to solve a specific problem. For the case of orbital integration, the important factors are ease of implementing the basic algorithm, core storage requirements, procedures for starting the process, interpolation, local error estimation and control, and a variable stepsize process. Furthermore, varying the physical model and using "non-smooth" effects (drag, solar radiation pressure, thrust, etc.) in the disturbing function are frequently required in programs which integrate orbits, so that the performance of the integration under these conditions should also be considered.

INTEGRATION PROGRAMS AND METHODS

The programs used for the numerical comparisons were the stand-alone Cowell orbit generator from the GEOSTAR system as documented in reference [4] and the stand-alone power series orbit generator from the NAP system as documented in reference [5]. Some of the basic characteristics of these programs pertinent to the comparisons are:

GEOSTAR

(i) Second sum ordinate Adams-Cowell predict, partial correct integration for the equations of motion;

(ii) Corrector-only integration for the variational equations;

(iii) Variable order, variable stepsize error control with independent step and order control for the equations of motion and variational equations integrations;

(iv) Geopotential model includes full 15×15 field;

(v) Variational equations for position partials with respect to initial state, geopotential coefficients, drag and radiation pressure parameters;

(vi) Third body lunar-solar gravitational effects with internally generated moon-sun ephemeris.

NAP

(i) Power series integration of equations of motion and variational equations.

(ii) Variable order, variable stepsize error control with independent order control for the equations of motion and variational equations integrations.

(iii) Geopotential model includes full 15×15 field.

(iv) Variational equations for position partials with respect to initial state, geopotential coefficients, drag, radiation pressure and mascon parameters. (Only state partials available in current version.)

(v) N-body gravitational field with planetary ephemeris series initialized using JPL ephemeris data.

Before presenting the numerical results obtained by using these programs, the following relative advantages and disadvantages are immediate concerning the two methods.

ADVANTAGES OF POWER SERIES INTEGRATION

(i) Assuming the order of the series can be made arbitrarily large, the integration error can be made arbitrarily small within the radius of convergence of the series, i.e., method enjoys the accuracy advantage of high order "multi-step" methods.

(ii) An integration step in the series method does not rely on a history of ordinate values, as do multistep methods. This implies that power series

integration enjoys the advantages of a "single-step" method in that it is self-starting and can dynamically change stepsize during the process without any significant effort.

(iii) The local error induced by the truncated power series can be readily estimated, allowing a simple error control algorithm.

(iv) Interpolation within the radius of convergence can be effected by a simple evaluation of the series.

DISADVANTAGES OF POWER SERIES INTEGRATION

(i) The method is not "general purpose" in the sense that each new problem to be solved requires either a new program or a significantly modified existing one.

(ii) The method is frequently difficult or impossible to implement since the function being integrated has to be expanded in a time series. Such a series may not exist, and even if it does, it may not be possible to derive analytically.

(iii) A program using the power series method is frequently not amenable to small changes in the function being integrated, making it expensive to maintain as a general tool for research and development.

(iv) The method requires that the function being integrated have high order continuous derivatives and hence is sensitive to small "non-smooth" anomalies.

ADVANTAGES OF COWELL INTEGRATION

(i) High order accuracy, limited only by numerical stability, can be easily attained.

(ii) The method is "general purpose" in the sense that many problems could be solved with essentially the same program.

(iii) The method is simple to implement, requiring only knowledge of the function being integrated. Furthermore, since it is nearly independent of this function, changes or modifications can be readily made.

(iv) The method is insensitive to small "non-smooth" anomalies in the function being integrated.

(v) The local error can be readily estimated.

DISADVANTAGES OF COWELL INTEGRATION

(i) The method is not self-starting, requiring an independent procedure to start.

(ii) The method requires, at each step, a "history" of ordinate values requiring an independent procedure to restart, after a step change.

(iii) Interpolation has to be done by an independent method.

(iv) The method is sensitive to high order numerical instabilities, effectively restricting the order of the method.

An implication one can derive from these considerations is that if the function to be integrated is fairly simple, analytic or very smooth, and not subject to frequent modifications, then power series is probably the best technique to use, if all the necessary derivatives can be easily computed. A non trivial application which satisfies these conditions is the restricted three body problem, and the power series method has been used quite successfully for this case; see for example references [6] and [3].

In contrast, the application of power series to the problem of the motion of an artificial satellite governed by a non-spherical central body together with third body effects, radiation pressure and atmospheric drag forces is not as clear. The problem generally has a very complex formulation, is subject to discontinuities and the model is constantly being modified or changed as our knowledge of the nature of the physical forces increases or the need for more accurate modeling arises.

NUMERICAL RESULTS

To try to determine the relative effectiveness of the two methods for this problem, the NAP and GEOSTAR orbit generator programs were used to integrate the equation for the force due to a nonspherical central body, i.e.,

$$\ddot{\bar{x}} = -\frac{GM}{r^3}\,\bar{x} + \frac{\partial U}{\partial \bar{x}}, \tag{1}$$

where

$$U = \frac{GM}{r}\left[\sum_{n=2}^{15}\sum_{m=0}^{n}\left(\frac{R_e}{r}\right)^n (c_n^m \cos m\lambda + s_n^m \sin m\lambda)\, P_n^m(\sin\psi)\right],$$

r = radius from center of earth to satellite,

R_e = earth's semi-major axis,

GM = gravitational constant times mass of earth,

λ = geocentric longitude (positive east),

ψ = geocentric latitude,

$P_n^m(\sin\psi)$ = associated Legendre polynomials.

Differences in the way these programs handle planetary ephemeris, solar radiation, and drag did not allow the inclusion of these effects in the comparisons. However, it is felt that the above problem is sufficiently representative to give useful results.

The following remarks are made in reference to the numerical results:

(1) All integrations were made with both the GEOSTAR and NAP programs. The integrations were performed with a full 15th order gravity model. An explanation of the various columns in tables 1-6 appears as follows: In the case of the NAP integrations the columns from left to right have the following meanings:

P_1/P_2 - number of terms used in integration of equations of motion and variational equations respectively; (P_2 = 0 indicates no variational equations are integrated).

N - Truncation error exponent, with allowable truncation error given by $0.5 \times 10^{-N} \times R$ (see Equation (8));

Δh - range of integration step sizes used (power series integration was found to be most efficient with a variable stepsize mode);

Δx - Maximum modulus of position error, in meters, from a reference solution;

$\Delta \dot{x}$ - Maximum modulus of velocity error, in meters per second;

$\Delta X(t)$ - number of digits in position partial $\frac{\partial x(t)}{\partial x_0}$ which agree with the reference solution;

CPU - Central processing unit time in minutes.

Similarly, in the case of the GEOSTAR integrations the first two columns from left to right are given by

Table 1

ATS-1 – 90 Day Arc – Full 15 × 15 Potential – 360/75

NAP – Truncation Error 0.5×10^{-N}							GEOSTAR (Fixed Step)					
P_1/P_2	N	Δh (sec)	Δx m	$\Delta \dot{x}$ m/s	$\Delta X(t)$	CPU Time (min)	P_1/P_2	h (sec)	Δx m	$\Delta \dot{x}$ m/s	$\Delta X(t)$	CPU Time (Min)
16/12	11	13600–14200	6.6	.0005	3	30.1	11/7	1800	7.4	.0005	4	6.0

ATS-1 ORBITAL DATA

$a = 42166307.5$ Meters

$e = 0.000378974$

$I = 1.881209$ Degrees

$\Omega = 83.365985$ Degrees

$\omega = 110.565390$ Degrees

$M = 351.771311$ Degrees

Table 2

Fictitious Satellite – 90 Day Arc – Full 15 × 15 Potential – 360/75

NAP – Truncation Error 0.5×10^{-N}							GEOSTAR (Fixed Step)					
P_1/P_2	N	Δh (Secs)	Δx m	$\Delta \dot{x}$ m/s	$\Delta X(t)$	CPU Time (Min)	P_1/P_2	h (Sec)	Δx m	$\Delta \dot{x}$ m/s	$\Delta X(t)$	CPU Time (Min)
16/12	15	11,000 - 24,500		—Reference—		20.7	11/7	5,000		—Reference—		2.1
16/12	11	21,000 - 45,000	1.5	0.9×10^{-5}	4	11.4	11/7	9,000	0.5	0.2×10^{-6}	5	1.6
16/12	10	26,000 - 52,500	1.8	0.3×10^{-5}	3	9.8	11/7	10,000	5.5	0.3×10^{-4}	4	1.5
16/0	11	21,000 - 45,000	1.5	0.9×10^{-5}	-	8.3	11/0	9,000	0.5	0.2×10^{-6}	-	1.0
16/0	10	26,000 - 52,500	1.8	0.3×10^{-5}	-	7.6	11/0	10,000	5.5	0.3×10^{-4}	-	1.1

FICTITIOUS ORBITAL DATA

a = 232358456.5 Meters — Ω = 0.0 Degrees

e = 0.377182500 — ω = 180.001343 Degrees

I = 45.0 Degrees — M = 179.998287 Degrees

Table 3

GEOS-II – 2 Day Arc – Elliptic Motion – 360/75

NAP – Truncation Error 0.5×10^{-N}							GEOSTAR (Fixed Step)					
P_1/P_2	N	Δh (sec)	Δx m	$\Delta \dot{x}$ m/s	$\Delta X(t)$	CPU Time (Min)	P_1/P_2	h (sec)	Δx m	$\Delta \dot{x}$ m/s	$\Delta X(t)$	CPU Time (Min)
16/12	15	275 - 455	Reference			2.8	11/7	50	Reference			1.9
16/12	10	600 - 950	4.5	.004	3	1.4	11/7	200	2.2	.004	3	0.7
16/12	9	690 - 1200	33.3	.03	3	1.2	11/7	225	27.5	.02	3	0.7
14/12	10	460 - 755	1.5	.002	4	1.6	11/9	175	1.9	.003	3	0.8
16/10	10	600 - 950	4.5	.004	2	1.1	9/6	125	3.8	.005	4	0.9
16/8	10	600 - 950	4.5	.004	0	1.0						
16/12	–	750	2.4	.003	5	1.3						
16/12	–	775	5.3	.006	5	1.3						
16/12	–	800	160.6	.1	4	1.2						

GEOS II ORBITAL DATA

a = 8070920.2 Meters

e = 0.071910451

I = 59.380155 Degrees

Ω = 337.265399 Degrees

ω = 185.153717 Degrees

M = 237.671006 Degrees

Table 4

GEOS-II – 2 Day Arc – Full 15 × 15 Potential – 360/75

NAP – Truncation Error 0.5×10^{-N}							GEOSTAR (Fixed Step)					
P_1/P_2	N	Δh (sec)	Δx m	$\Delta \dot{x}$ m/s	$\Delta X(t)$	CPU Time (Min)	P_1/P_2	h (sec)	Δx m	$\Delta \dot{x}$ m/s	$\Delta X(t)$	CPU Time (Min)
16/12	15	165 - 280		— Reference —			11/7	50	— Reference —			3.4
16/12	11	270 - 580	0.2	.001	4	23.0	11/9	200	3.8	.001	4	1.4
16/12	10	340 - 670	4.0	.004	2	18.4	11/7	225	36.5	.03	4	1.3
16/14	10	340 - 670	4.0	.004	4	21.9	11/7	175	3.2	.004	4	1.7
16/12	9	410 - 770	170	0.4	2	15.1	11/7	150	1.1	.001	5	1.7
16/12	–	375	3.0	.002	4	23.0						

GEOS II ORBITAL DATA

a = 8070920.2 Meters	Ω = 337.265399 Degrees
e = 0.071910451	ω = 185.153717 Degrees
I = 59.380155 Degrees	M = 237.671006 Degrees

Table 5

IMP–IV – 4 Day Arc – Elliptic Motion – 360/75

NAP – Truncation Error 0.5×10^{-N}							GEOSTAR – Truncation Error 0.25×10^{-N}						
P_1/P_2	N	Δh (Secs)	Δx m	$\Delta \dot{x}$ m/s	$\Delta X(t)$	CPU Time (Min)	P_1/P_2	N	Δh (Secs)	Δx m	$\Delta \dot{x}$ m/s	$\Delta X(t)$	CPU Time (Min)
16/12	15	100 - 24,000	—— Reference ——			0.5	11/7	7	24 - 2,520	—— Reference ——			0.5
16/9	11	190 - 45,000	2.0	.0005	4	0.3	11/7	5	24 - 3,600	1.4	.0002	7	0.4
16/9	10	220 - 50,000	13.0	.001	4	0.3	11/7	4	40 - 4,560	3.7	.00007	6	0.4

IMP–IV ORBITAL DATA

a = 111[illegible]7034.5 Meters	Ω = 167.877476 Degrees
e = 0.940607086	ω = 180.020584 Degrees
I = 67.140330 Degrees	M = 0.011962 Degrees

Table 6

IMP-IV – 4 Day Arc – Full 15 × 15 Potential – 360/75

NAP – Truncation Error 0.5×10^{-N}							GEOSTAR – Truncation Error 0.25×10^{-N}						
P_1/P_2	N	Δh (secs)	Δx m	$\Delta \dot{x}$ m/s	$\Delta X(t)$	CPU Time (Min)	P_1/P_2	N	Δh (Secs)	Δx m	$\Delta \dot{x}$ m/s	$\Delta X(t)$	CPU Time (Min)
16/12	15	85 - 19,500		—Reference—		3.7	11/7	7	24 - 2,520		—Reference—		0.8
16/9	12	120 - 31,600	0.8	.0003	4	2.3	11/7	5	24 - 3,420	1.4	.0004	6	0.7
16/9	11	150 - 35,000	0.4	.002	4	2.3	11/7	4	40 - 4,680	41.3	.0001	7	0.6
							11/7	6	24 - 2,460	0.2	.00005	6	0.7

IMP-IV ORBITAL DATA

a = 111607034.5 Meters	Ω = 167.877476 Degrees
e = 0.940607086	ω = 180.020584 Degrees
I = 67.140330 Degrees	M = 0.011962 Degrees

P_1/P_2 - represent orders in predictor corrector integration formulas for equations of motion and variational equations respectively, ($P_2 = 0$ indicates no variational equations are integrated)

h - integration step in seconds.

The remaining columns have meanings as already denoted above.

(2) The three satellites ATS-1, GEOS-B, IMP-IV, plus a fictitious satellite were selected for the integration with 2, 4, and 90 day arcs.

(3) The computations were performed on the IBM 360 model 75 and were run in a "sequential processing" mode so that the CPU times per run are representative of the calculation time required by each process.

(4) The term "elliptic motion" indicates that the integration was performed with a point mass 2-body acceleration only, i.e., U in Equation (1) was set to zero.

An immediate observation concerning these numerical results is that for the integration of a near-earth circular satellite such as GEOS-B (Table 4), the power series method was 10-15 times more expensive to use, and that for satellites which are further away, such as ATS-A or the fictitious satellite given in table 2, this ratio of power series time to Cowell time drops to 5 or 6. Such results indicate the sensitivity of the power series to the non-smooth perturbative effects of the geopotential. This is again exemplified in tables 5 and 6, where for the highly elliptic IMP-F case, the two body runs show power series superior to Cowell, but the addition of the perturbative potential increases the power series/Cowell ratio from less than 1 to 2 or 3. To see why the power series method is slower although it allows larger stepsizes, the following sections present an analysis of the two methods as applied to the two body problem. Although this analysis does not reflect the actual computations performed using the geopotential, it details how these methods are constructed and the work required to perform an integration step.

POWER SERIES SOLUTION FOR TWO BODY EQUATIONS OF MOTION

We wish to solve the two body problem

$$\ddot{\bar{x}} = -\frac{G\,M}{r^3}\,\bar{x} \tag{2}$$

with initial conditions $\bar{x}_0$ and $\dot{\bar{x}}_0$ given at time t_0.

The position of the satellite requires the following series with K terms

$$r(t) = \sum_{j=0}^{K} R_j (t - t_0)^j$$

$$\bar{x}(t) = \sum_{j=0}^{K} \bar{R}_j (t - t_0)^j \qquad (3)$$

where

$$\bar{x}_0 = \bar{R}_0$$

$$\dot{\bar{x}}_0 = \bar{R}_1$$

$$|\bar{x}_0| = R_0$$

$$|\dot{\bar{x}}_0| = R_1$$

and series required for the determination of velocity are given by

$$\dot{r}(t) = \sum_{j=1}^{K} j\, R_j (t - t_0)^{j-1}$$

$$\dot{\bar{x}}(t) = \sum_{j=1}^{K} j\, \bar{R}_j (t - t_0)^{j-1} \qquad (4)$$

To determine the coefficients $\bar{R}_i$, two auxiliary functions are introduced:

$$\theta(t) = -\frac{G\,M}{r^3(t)} = \sum_{j=0}^{K} \Theta_j (t - t_0)^j$$

$$
w(t) = \frac{\dot{r}(t)}{r(t)} = \sum_{j=0}^{K} w_j (t - t_0)^j
$$

and using these functions, the coefficients for the required series are determined recursively as follows:

$$
R_K = \frac{1}{2R_0} \left[2\, \bar{R}_K^T \bar{R}_K + \sum_{j=1}^{K-1} \left(\bar{R}_j^T \bar{R}_{K-j} - R_j R_{K-j} \right) \right],
$$

$$
w_{K-1} = \frac{1}{R_0} \left[K R_K - \sum_{j=1}^{K-1} R_j w_{K-1-j} \right],
$$

$$
\Theta_K = -\frac{3}{K} \sum_{j=0}^{K-1} \Theta_j w_{k-1-j},
$$

$$
\bar{R}_{K+2} = \frac{1}{(K+2)(K+1)} \sum_{j=0}^{K} \Theta_j \bar{R}_{K-j}.
$$

At each integration step there are 9K + 6 multiplications, 8K additions, and 6 divisions to obtain the coefficients R_K, $\bar{R}_K$ in the power series.

The number of operations involved in obtaining $\bar{x}(t)$ from its series representation as given above is 3K multiplications and 3K additions. The velocity vector $\dot{\bar{x}}(t)$ requires 3(2K - 1) multiplications and 3 (K - 1) additions.

Hence, approximately 32K + 6 operations are required for integrating the two body equations of motion from time t to time t + h. The selection of K = 16 then gives a total of 518 operations.

POWER SERIES SOLUTION FOR THE VARIATIONAL EQUATIONS OF TWO BODY MOTION

The acceleration for unperturbed motion is given by the vector equation

$$\ddot{\bar{x}} = -\frac{G\,M}{r^3}\,\bar{x}.$$

The state transition matrix relating variations in position at time t to position at time t_0 is given by the first and second fundamental matrix solutions of the variational system

$$\ddot{y} = (\theta\,I + \tilde{\bar{x}}\,\bar{x}^T)\,(t)\,y = A\,(t)\,y \tag{5}$$

where

$\theta = -\,G\,M\;r^{-3}$, $\tilde{\bar{x}} = \varphi\,\bar{x}$, $\varphi = 3\;G\,M\;r^{-5}$,

$I = 3 \times 3$ identity matrix.

The 6×6 state transition is

$$X\,(t,\,t_0) = \begin{bmatrix} U & V \\ \dot{U} & \dot{V} \end{bmatrix},$$

where $U\,(t,\,t_0)$ and $V\,(t,\,t_0)$ are 3×3 matrices satisfying (5) with initial conditions

$$U(t_0,\,t_0) = I,\; V(t_0,\,t_0) = 0$$

$$\dot{U}(t_0,t_0) = 0,\; \dot{V}\,(t_0,\,t_0) = I.$$

In addition to the auxiliary functions $\theta(t)$ and $w(t)$, which were introduced previously for the equations of motion, another auxiliary function $\varphi(t)$ given by

$$\varphi(t) = \frac{3\,G\,M}{r^5(t)} = \sum_{j=0}^{K} \Phi_j\,(t - t_0)^j$$

is used for the variational equations.

The required series are given by

$$\tilde{\bar{x}}(t) = \varphi\,\bar{x}(t) = \sum_{j=0}^{K} \tilde{\bar{R}}_j\,(t - t_0)^j$$

$$U(t, t_0) = \sum_{j=0}^{K} U_j\,(t - t_0)^j \tag{6}$$

$$V(t, t_0) = \sum_{j=0}^{K} V_j\,(t - t_0)^j$$

which are needed together with the series for $\bar{x}(t)$ given previously.

The coefficients in these series are obtained in a recursive manner as follows

$$\Phi_K = -\frac{5}{K}\sum_{j=0}^{K-1} \Phi_j\,w_{K-1-j} \quad \text{(series for } \varphi)$$

$$\tilde{\bar{R}}_K = \sum_{j=0}^{K} \phi_j \bar{R}_{K-j} \quad \text{(series for } \tilde{\bar{x}} \text{ column matrix)}$$

$$A_K = \Theta_K I + \sum_{j=0}^{K} \tilde{\bar{R}}_j \bar{R}^T_{K-j} \quad \text{(series for } A_{3\times 3} \text{ symmetric matrix)}$$

$$U_{K+2} = \frac{1}{(K+1)(K+2)} \sum_{j=0}^{K} A_j U_{K-j} \quad \text{(series for } U_{3\times 3}\text{)}$$

$$V_{K+2} = \frac{1}{(K+1)(K+2)} \sum_{j=0}^{K} A_j V_{K-j} \quad \text{(series for } V_{3\times 3}\text{)}.$$

Hence we have

$$X(t, t_0) = \left[\begin{array}{c:c} \displaystyle\sum_{j=0}^{K} U_j (t-t_0)^j \quad {}_{3\times 3} & \displaystyle\sum_{j=0}^{K} V_j (t-t_0)^j \quad {}_{3\times 3} \\ \hdashline \displaystyle\sum_{j=1}^{K} j\, U_j (t-t_0)^{j-1} \quad {}_{3\times 3} & \displaystyle\sum_{j=1}^{K} j\, V_j (t-t_0)^{j-1} \quad {}_{3\times 3} \end{array}\right]. \tag{7}$$

Each integration step is found to require approximately 67K + 68 multiplications, 49K + 4 additions, and 2 divisions or approximately 116K + 74 operations for the U_j, V_j coefficients in the above series.

The number of operations required to obtain the matrices $U_{3\times3}$ and $V_{3\times3}$ is 18K multiplications and 18K additions. The derivatives $\dot{U}$ and $\dot{V}$ require a total of 36K - 18 multiplications and 36K - 18 additions. The sum of all the above operations is approximately 224K + 38 operations. The selection of K = 12 yields approximately 2726 operations for obtaining the position and velocity partials by the power series method.

Power Series Variable Stepsize Integration

Consider the power series for position

$$\bar{x}(t) = \sum_{j=0}^{K} \bar{R}_j (t - t_0)^j .$$

Assume the modulus of the last term in the series is required to be equal to some truncation parameter ϵ. Then the appropriate step size may be computed by solving

$$\bar{R}_k \cdot \bar{R}_k (t - t_0)^{2K} = \epsilon^2,$$

i.e.,

$$h = t - t_0 = \left(\frac{\epsilon}{|\bar{R}_k|}\right)^{1/K} . \qquad (8)$$

In NAP, the truncation parameter ϵ is chosen to be $0.5 \times 10^{-N} \times R$, where

N is a given truncation error exponent,

R is the modulus of $\bar{x}(t)$.

COWELL INTEGRATION OF THE EQUATIONS OF MOTION

Predictor-Corrector Integration Formulas

The integration formulas used for integrating the equations of motion are given by [4]:

$$\bar{x}_{n+1} = h_e^2 \left[{}^{II}\bar{S}_n + \sum_{i=0}^{K_e} \alpha_i \ddot{\bar{x}}_{n-i} \right] \text{(Störmer predictor)}$$

$$\bar{x}_{n+1} = h_e^2 \left[{}^{II}\bar{S}_n + \sum_{i=0}^{K_e} \alpha_i^* \ddot{\bar{x}}_{n+1-i} \right] \text{(Cowell corrector)}$$

$$\dot{\bar{x}}_{n+1} = h_e \left[{}^{I}\bar{S}_n + \sum_{i=0}^{K_e} \beta_i \ddot{\bar{x}}_{n-1} \right] \text{(Adams predictor)} \tag{9}$$

$$\dot{\bar{x}}_{n+1} = h_e \left[{}^{I}\bar{S}_n + \sum_{i=0}^{K_e} \beta_i^* \ddot{\bar{x}}_{n+1-i} \right] \text{(Moulton corrector)}$$

where

h_e = the stepsize used for the integration of the equations of motion,

K_e = $p - 2$, where p is the order of the formulas,

${}^{I}\bar{S}_n$, ${}^{II}\bar{S}_n$ = the first and second sums of the accelerations,

α_i, α_i^*, β_i, β_i^* = the integration coefficients for the ordinate form,

${}^{I}\bar{S}_n = \nabla^{-1} \ddot{\bar{x}}_n$,

${}^{II}\bar{S}_n = \nabla^{-1}\, {}^{I}\bar{S}_n = \nabla^{-2} \ddot{\bar{x}}_n$.

Algorithm for Integrating the Equations of Motion

The procedure for integrating the equations of motion is as follows:

(i) Obtain a set of "starting" values for the accelerations

$$\ddot{x}_{K_e - i}, \quad i = 1, 2, \ldots, K_e$$

and sums ${}^{I}\bar{S}_{K_e}$, ${}^{II}S_{K_e}$ using an independent procedure;

(ii) Obtain predicted values of $\bar{x}^{(P)}_{n+1}$ and $\dot{\bar{x}}^{(P)}_{n+1}$ using the Störmer and Adams predictor formulas with $n = K_e$;

(iii) Compute the acceleration $\ddot{\bar{x}}_{n+1}$ using the last predicted position and velocity vector;

(iv) Obtain corrected values of $\bar{x}^{(c)}_{n+1}$ and $\dot{\bar{x}}^{(c)}_{n+1}$ using the Cowell and Moulton corrector formulas;

(v) Compute the magnitude of the vector

$$\left| \bar{x}^{(c)}_{n+1} - \bar{x}^{(P)}_{n+1} \right|$$

and compare with a predictor-corrector tolerance. A predictor-corrector cycle is completed when this vector magnitude is sufficiently small. A maximum number of three iterations is allowed.

(vi) Compute new first and second sums by

$$\begin{aligned} {}^{I}\bar{S}_{n+1} &= \ddot{\bar{x}}_{n+1} + {}^{I}\bar{S}_{n} \\ {}^{II}\bar{S}_{n+1} &= {}^{I}\bar{S}_{n+1} + {}^{II}\bar{S}_{n}. \end{aligned} \tag{10}$$

This completes one integration step and the integration is continued by repeating steps (ii) through (vi) with n replaced by n + 1.

In the case in which the accelerations acting on the satellite do not depend on the velocity, the velocity vector is not predicted, and it is computed only after convergence of the position is obtained by the Cowell Corrector.

The number of mathematical operations required in a single step of Cowell integration of the equations of motion is determined from Equations (9) and (11). Each Cowell integration step requires approximately 24 K_e + 26 operations. The selection of K_e = 9 corresponding to order 11 gives a total of 242 operations.

The ratio for the number of operations of power series to Cowell is approximately 518/242, or about 2/1. The introduction of perturbative effects in the

equations of motion will, of course, greatly increase the relative number of operations in the power series method because of the additional higher order derivatives which are required. These results explain, partially, the relative inefficiency of the power series method even though it generally allows larger stepsizes.

Variable Stepsize Integration

Stepsize modification is designed specifically for the integration of the equations of motion and the concomitant variational equations associated with highly eccentric orbits and where fixed stepsize integration is extremely inefficient.

The variation in stepsize is based on the concept of local error control by means of the following constraint equation

$$T_2 \leq R_n \leq T_1 \ , \tag{11}$$

where T_1 and T_2 are specified error bounds. If the local error satisfies $R_n > T_1$, the stepsize is decreased, and if $R_n > T_2$, the stepsize is increased. The local error is estimated by the formula

$$R_n \cong c \ \left| \bar{x}_n^P - \bar{x}_n^c \right| ,$$

where c is a constant and $\bar{x}_n^P$, $\bar{x}_n^c$ are the predicted and corrected values of the position vectors. The stepsize is computed from the equation

$$h_{new} = h_{old} \left(\frac{T_3}{R_n} \right)^{1/K} , \tag{12}$$

where T_3 is a specified value for the local error in the range $T_2 \leq T_3 \leq T_1$, and where K is the number of backpoints used in the integration.

COWELL INTEGRATION OF THE VARIATIONAL EQUATIONS

The position and velocity partials as given in reference [4] for velocity free accelerations can be expressed as

(letting $[U(t_n, t_0), V(t_n, t_0)]_{3\times 6} = [U, V]_n$),

$$[U, V]_{n+1} = [I - H]^{-1} [W_n]_{3\times 6} \tag{13}$$

$$[\dot{U}, \dot{V}]_{n+1} = [h_v \beta_0^* A_{n+1}] [U, V]_{n+1} + [Y_n]_{3\times 6}$$

where

$$H = h_v^2 \alpha_0^* A_{n+1}$$

$$W_n = h_v^2 \left[{}^{II}P_n + \sum_{j=1}^{K_v} \alpha_j^* [\ddot{U}, \ddot{V}]_{n+1-j} \right]$$

$$Y_n = h_v \left[{}^{I}P_n + \sum_{j=1}^{K_v} \beta_j^* [\ddot{U}, \ddot{V}]_{n+1-j} \right]$$

h_v = the step size used for integrating the variational equations,

K_v = p - 2, where p is the order of the integration formulas used for the variational equations,

${}^{I}P_n$, ${}^{II}P_n$ = the first and second sums 3×6 matrices of the acceleration partials,

α_j^*, β_j^* = the ordinate form of the integration coefficients corresponding to K_v.

The inversion of the matrix (I - H) when performed by Gaussian elimination requires $n^3/3 + n^2 - n/3$ multiplications and divisions, where n is the order of the matrix [7]. For the case considered here, n = 3, and therefore about 17 operations are required for the inversion. The calculation of position and velocity partials by Cowell integration requires approximately 72 K_v + 342 operations. For K_v = 5, corresponding to order 7, this gives a total of 702 operations. The ratio for operations of power series to Cowell for the variational equations is approximately 2726/702 or about 4/1.

SUMMARY AND CONCLUSIONS

The power series and Cowell methods of integration have been compared, and it has been shown that although the power series is a highly stable, accurate technique for orbital integration, it is considerably more expensive to use and maintain compared to the Cowell method when applied to the problem of integrating highly perturbed orbital motion. On the other hand, power series integration affords a very accurate analytical tool which may offer significant advantages for error analysis work or the integration of low perturbed orbital motion such as transfer trajectories.

REFERENCES

1. Henrici, P., "Discrete Variable Methods in Ordinary Differential Equations," J. Wiley, New York, 1962.

2. Steffensen, J. F., "On the Restricted Problem of Three Bodies," Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd, 30, No. 18, 1956.

3. Fehlberg, E., "Numerical Integration of Differential Equations by Power Series Expansions, Illustrated by Physical Examples, NASA TN D-2356, Oct. 1964.

4. Velez, C., Brodsky, G., "GEOSTAR-I A Geopotential and Station Position Recovery System," NASA X Report X553-69-544, Nov. 1967.

5. Hartwell, J. G., "Mathematical and Programming Documentation of a General Purpose Integration Using Power Series Methods," DBA report prepared for NASA Langley under contract number NAS1-9389, Dec. 1969.

6. Deprit, H. and Price J. F., "The Computation of Characteristic Exponents in the Planar Restricted Problem of Three Bodies," The Astro. J. Vol. 70, No. 11, Dec. 1965, pp. 836-846.

7. Faddeeva, V. N., "Computational Methods of Linear Algebra," Dover, 1959.